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ON MATERIAL SYMMETRY IN MECHANICS

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Abstract—The main feature of this paper is the usage of transformations of all coordinates present in the constitutive equation of a given class of materials (generalised coordinates) to describe symmetry of materials within the class. The materials considered here are elastic materials and liquid crystals, although the procedure can be used to describe the symmetry of other classes.

One of the end-results of the exercise is that various theories of symmetry can be viewed together.

1. INTRODUCTION

The theory of symmetry has been formulated in the literature for many materials in mechanics. For instance, the case of simple materials is formulated by Truesdell in ([1], Chap. IV) and that of shells is discussed by Ericksen in [2]. Material symmetry is also discussed by Coleman [3], Wang [4] and Carroll and Naghdi [5].

Normally, materials are classified according to their constitutive equations. For example, a material for which the stress at each of its points depends only on the local deformation gradient there is called *elastic*. Within any class are some materials which exhibit special symmetries, e.g. some elastic materials, called *isotropic solids* cannot detect any rotation or translation of the reference configuration or, in other words, of the material coordinates. This symmetry of an isotropic solid within the class of elastic materials is therefore characterised by the group of all rotations and translations.

To cover more special cases, the group of all unimodular linear transformations of material coordinates is used [see 1] to describe symmetry within the class of elastic materials. However, if symmetry transformations are restricted to transformations of material coordinates, the symmetry of liquid crystals is not adequately described in an appropriate class of orientable materials. To properly describe this symmetry, one needs to use a larger group. We seek an approach that unifies these theories of symmetry.

We consider transformations of all generalized coordinates present in a given constitutive equation in order to describe the symmetry within the class specified by the equation. This set of transformations well describe the symmetry of liquid crystals and, of course, it includes the group of transformations in [1], when applied to elastic materials. In Section 2, we list some requirements usually made on symmetry transformations. These are physically motivated. Transformations which are obtained for the materials considered are also listed in Section 2. One of the requirements on the symmetry transformations is that the representation of frame indifference for a constitutive equation be preserved. This eventually leads to our seeking all the automorphisms of the group of orientation—preserving isometries of Euclidean space. The result obtained is given in Section 3. In Section 4, we put these ideas together to obtain limitations on the mappings available.

2. REQUIREMENTS ON SYMMETRY TRANSFORMATIONS

Let us assume that we are given a constitutive equation of the form

$$W = \hat{W}(v_i(u^p), u^q),$$

for $1 \le i \le l$, $1 \le p \le m$, $1 \le q \le m$, where for a point $(u^1, u^2, \ldots u^m)$, W depends only on the values of the functions $v^i(u^p)$ in an infinitesimal neighborhood of $(u^1, u^2, \ldots u^m)$. In this equation, u^p, $1 \le p \le m$ are real variables, $v_i(u^p)$, $1 \le i \le l$ are real-valued functions, and W is the energy per unit coordinate volume, the volume measure being

$$\mathrm{d}u = \mathrm{d}u^1\,\mathrm{d}u^2 \ldots \mathrm{d}u^m.$$

As indicated above, W denotes the values assumed by the function $\hat{W}$. We shall consider the special form

$$W = \hat{W}\left(\frac{\partial v_i}{\partial u^p}, v_j, u^q\right), \tag{2.1}$$

where W at point $(u^1, u^2, \dots u^m)$ now depends on $(\partial v_i/\partial u^p, v_i, u^q)$ at that point. The set of materials having this type of constitutive equation includes elastic materials, shells and static theories of liquid crystals.

As stated in the introduction, we shall start with a set of transformations of the generalized coordinates. With some smoothness and invertibility requirements, these will induce transformations of the constitutive equations, our goal being to describe the symmetry of a body. We shall now put this in formal terms. Let $\mathscr{D}'$ be the set of all maps

$$\mathbf{T}': \mathcal{O} \times \mathscr{I} \to (\mathbf{R}^l \times \mathbf{R}^m) \times \mathbf{R}^1$$
$$(\mathbf{v}, u, W) \mapsto (\mathbf{T}_1'(\mathbf{v}, u), T_2'(\mathbf{v}, u), T_3'(\mathbf{v}, u, W)) \equiv (\bar{\mathbf{v}}, \bar{u}, \bar{W})$$

where
$$\mathcal{O} \subset \mathbf{R}^l \times \mathbf{R}^m, \mathscr{I} \subset \mathbf{R},$$
$$u \equiv (u^1, u^2, \dots u^m),$$
$$\mathbf{v} \equiv (v_1, v_2, \dots v_l),$$
and
$$\frac{\partial \mathbf{v}}{\partial u} \equiv \left[\frac{\partial v_i}{\partial u^p}\right],$$

if v is a given function of u. We shall assume the following:

(i) $\exists u_0 \in \mathbf{R}^m \ni$ for every v which is the first component of an element in $\mathcal{O}$, $\{\mathbf{v}\} \times N(u_0) \subset \mathcal{O}$, $N(u_0)$ being a neighbourhood of u_0,

(ii) T′ is differentiable in all coordinates; (2.2)

(iii) T′ is 1–1, the inverse of T′ defined on its image satisfies (i) and (ii), and $(\mathbf{T}_1', T_2')$: $\mathcal{O} \to \mathbf{R}^l \times \mathbf{R}^m$ is also 1–1.

We associate with T′ the following transformation T of the constitutive eqn (2.1) at point u_0:

$$\mathbf{T}: \left(\left(\frac{\partial \mathbf{v}}{\partial u}(u_0), \mathbf{v}(u_0), u_0\right), W\right) \mapsto$$
$$\left(\left(\frac{\partial \bar{\mathbf{v}}}{\partial \bar{u}}(\bar{u}_0), \bar{\mathbf{v}}(\bar{u}_0), \bar{u}_0\right), (W + A(u_0))\left\|\frac{\partial u}{\partial \bar{u}}(u_0)\right\|\right) \tag{2.3}$$

where $\|\mathbf{F}\|$ denotes the absolute value of the determinant of F. In other words, the constitutive relation which associates a state $(\partial \mathbf{v}/\partial u\,(u_0), \mathbf{v}(u_0), u_0)$ with a real number W is transformed to one which associates the state $(\partial \bar{\mathbf{v}}/\partial \bar{u}\,(\bar{u}_0), \bar{\mathbf{v}}(\bar{u}_0), \bar{u}_0)$ with the product by $\|\partial u/\partial \bar{u}\,(u_0)\|$ of the sum of the same real number W and $A\,(u_0)$, A, $\bar{\mathbf{v}}$, $\bar{u}$ being determined by T′. Let $\mathscr{D}$ be the set of all such T that arises from an element in $\mathscr{D}'$. In (2.3), for a given state $(\partial \mathbf{v}/\partial u\,(u_0), \mathbf{v}(u_0), u_0)$,

$$\left[\frac{\partial \bar{u}}{\partial u}\right]pq \equiv \frac{\partial \bar{u}^p}{\partial u^q}(u_0) = \frac{\partial \bar{u}^p}{\partial v_i}(\mathbf{v}, u)\bigg|_{u_0} \frac{\partial v_i}{\partial u^q}(u_0) + \frac{\partial \bar{u}^p(\mathbf{v}, u)}{\partial u^q}\bigg|_{u_0},$$

$$\left[\frac{\partial \bar{\mathbf{v}}}{\partial \bar{u}}\right]_{ip} \equiv \frac{\partial \bar{v}_i}{\partial \bar{u}^p} = \left[\frac{\partial \bar{v}_i(\mathbf{v}, u)}{\partial v_j}\bigg|_{u_0} \frac{\partial v_j}{\partial u^q}(u_0) + \frac{\partial \bar{v}_i(\mathbf{v}, u)}{\partial u^q}\bigg|_{u_0}\right]\left[\frac{\partial u}{\partial \bar{u}}(u_0)\right]_{qp}$$

$[\partial u/\partial \bar{u}\,(u_0)]$ being the inverse matrix of $[\partial \bar{u}/\partial u\,(u_0)]$. Our assumptions [2.2] make this meaningful.

Definitions

Two material points u_0, u_1 are said to be T-*equivalent* if $\exists$ a transformation $\mathbf{T} \in \mathscr{D} \ni$

$$T_2'(u_0) = u_1,\ \hat{W}\left(\frac{\partial \bar{\mathbf{v}}}{\partial \bar{u}}(u_1), \bar{\mathbf{v}}(u_1), u_1\right) = \left(\hat{W}\left(\frac{\partial \mathbf{v}}{\partial u}(u_0), \mathbf{v}(u_0), u_0\right) + A(u_0)\right)\left\|\frac{\partial u}{\partial \bar{u}}(u_0)\right\|, \tag{2.4}$$

for every state $(\partial\bar{v}/\partial\bar{u}\,(u_i), \bar{v}(u_1), u_1)$, $(\partial v/\partial u\,(u_0), v(u_0), u_0)$ being the corresponding state.

The constitutive eqn (2.1) is said to be *invariant* under **T** at a material point u_0 if u_0 is **T**-equivalent to itself. Our definition (2.3) of **T** presumes that

$$\bar{W} = (W + A(u_0))\left\|\frac{\partial u}{\partial \bar{u}}(u_0)\right\|$$

and that this is an appropriate law of transformation for energy densities. Actually, this transformation of W is suggested in[6] where quantities are transformed in such a way as to preserve some conservation laws. The energy density W can be identified with either $\rho^0 A$ or $\rho^0 B$ in[6]. The transformation rules given for these quantities reduce in our case to what we have used in (2.3).

Having introduced $\mathcal{D}$ as the set of transformations which we intend to use to describe symmetry within the class (2.1), we next indicate how one can use physical considerations to restrict $\mathcal{D}$ to a smaller set $\mathcal{S}$.

Our set of symmetry transformations $\mathcal{S}$ should satisfy the following:

(a) *Domain considerations.* The state space is the set of all $(\partial v/\partial u, v, u)$ that are physically admissible for the material under consideration. For example, they might include only states for which $(\partial v/\partial u)$ is of a certain rank. Every element of $\mathcal{S}$ should be such that $(\partial\bar{v}/\partial\bar{u}, \bar{v}, \bar{u})$ is also admissible. We give another example. In some theories, from physical considerations, the energy density does not have a constant value over a set of states. For instance, in elasticity theory, $\hat{W}$ does not usually assume a constant value on the set of all dilatations of a given state. We shall require that its transform under any symmetry transformation in $\mathcal{S}$ will not do so.

(b) *Preservation of frame indifference.* Normally, constitutive equations satisfy the principle of frame indifference. Every element of $\mathcal{S}$ should transform a given constitutive equation to another which is also frame indifferent. This issue will be discussed in Section 3.

If $\mathcal{S}_0 = \{\mathbf{T} \in \mathcal{D} | \bar{u}_0 = u_0\}$, $\mathcal{S}_0$ forms a group under composition. We note that it is possible that there may not be a material that has the whole of $\mathcal{S}_0$ as its group of symmetry. However, in such cases, one could list subgroups of $\mathcal{S}_0$ that are symmetry groups for some materials.

Some results

We shall now list the symmetry transformations that result if we apply some physical considerations on the set $\mathcal{D}$ for elastic materials and liquid crystals. The proofs are contained in Section 4.

Elasticity theory. The symmetry transformations are of the form

$$\mathbf{T}:\left(\left(\frac{\partial \mathbf{x}}{\partial X}, X_0\right), W\right) \mapsto \left(\left(\alpha \mathbf{R}_0 \frac{\partial \mathbf{x}}{\partial X}\frac{\partial X}{\partial \bar{X}}, \bar{X}_0\right), (W + A(X_0))\left\|\frac{\partial X}{\partial \bar{X}}\right\|\right)$$

where $\mathbf{R}_0$ is any rotation and

$$\alpha \neq 0, \alpha\left|\frac{\partial X}{\partial \bar{X}}\right| = \alpha\left(\det\frac{\partial X}{\partial \bar{X}}\right) > 0, \left\|\frac{\partial X}{\partial \bar{X}}\right\| = \left|\det\frac{\partial X}{\partial \bar{X}}\right|.$$

If restricted to $\|\partial\bar{X}/\partial X\| = 1$, $\alpha = 1$, $R_0 = 1$, and from the discussions in[7], the set reduces to that in[1].

Liquid crystals. In the class specified by

$$W = \hat{W}\left(\frac{\partial \mathbf{n}}{\partial X}, \frac{\partial \mathbf{x}}{\partial X}, \mathbf{n}, X\right) \tag{2.5}$$

where states satisfy, $\mathbf{n}\cdot\mathbf{n} = 1$, $|\partial\mathbf{x}/\partial X| = 1$, and $(\partial\mathbf{n}/\partial X)^T\mathbf{n} = 0$, the group $\mathcal{S}_{02}$ of all symmetry transformations with $\bar{u}_0 = u_0$ contains elements of the form

$$\mathbf{T}:\left(\left(\frac{\partial \mathbf{n}}{\partial X}, \frac{\partial \mathbf{x}}{\partial X}, \mathbf{n}, X_0\right), W\right) \mapsto \left(\left(\pm\mathbf{R}\frac{\partial \mathbf{n}}{\partial X}\frac{\partial X}{\partial \bar{X}}, \alpha\mathbf{R}\frac{\partial \mathbf{x}}{\partial X}\frac{\partial X}{\partial \bar{X}}, \pm\mathbf{R}\mathbf{n}, X_0\right), (W + A\,(X_0))\left\|\frac{\partial X}{\partial \bar{X}}(X_0)\right\|\right)$$

where $|\partial X/\partial \bar{X}| = (1/\alpha^3)$ and $\mathbf{R}$ is a rotation. The set of all elements in $\mathcal{S}_{02}$ that satisfies $\alpha = 1, A(X_0) = 0$, characterises the symmetry of the cholesteric liquid crystals.

As stated in the introduction, symmetry transformations for elastic materials are restricted in [1] to unimodular transformations of the material coordinates. In addition when discussions in [7] are also considered, frame indifference is taken to mean

$$\hat{W}\left(\mathbf{Q}\frac{\partial \mathbf{x}}{\partial X}, X_0\right) = \hat{W}\left(\frac{\partial \mathbf{x}}{\partial X}, X_0\right)$$

where $\mathbf{Q}$ is an orthogonal tensor and not just a rotation as assumed in this paper. If we do the same thing† for the class (2.5), the *most* symmetrical material similar to a liquid crystal will have [see Section 4] constitutive equation of the form

$$W = \hat{W}\left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}}, \mathbf{n}, X_0\right) = \hat{W}\left(\mathbf{Q}\frac{\partial \mathbf{n}}{\partial \mathbf{x}}\mathbf{Q}^T, \mathbf{Q}\mathbf{n}, X_0\right) \tag{2.6}$$

where $\mathbf{Q}$ is any orthogonal tensor. Since this includes symmetry under reflections and does not necessarily imply symmetry under reversal of sign of $\mathbf{n}$, it is clear‡ that unimodular transformations of material coordinates do not properly account for the symmetry of liquid crystals themselves within the class (2.5).

Material isomorphism

Definition. Two material points u_0, u_1 are said to be *materially isomorphic* if they are $\mathbf{T}$-equivalent for some $\mathbf{T} \in \mathcal{S}$.

We note that if any two points of a shell are isomorphic in the sense given above, the shell is uniform in the sense of definition (2.13) of [9].

3. FRAME INDIFFERENCE. SOME RESULTS IN GROUP THEORY

(a) *Frame indifference and theorem*

Although we have been concerned so far with a constitutive equation that relates energy density to the state of a body, it is clear that our discussions can be used to describe symmetry of constitutive equations that relate the state to other quantities such as a stress tensor. Here, we shall discuss constitutive equations in general terms as to include these cases.

Suppose we are given a constitutive equation denoted by f which relates states $\{\mathbf{s}\}$ to quantities $\{\mathbf{a}\}$, i.e.

$$\mathbf{s} \overset{f}{\longleftrightarrow} \mathbf{a}. \tag{3.1}$$

Generally, a transformation $\mathbf{T}$ will transform§ f to another constitutive equation $g = \mathbf{T}f$. Roughly, the principle of frame indifference will stipulate that a given material cannot distinguish between some frames. In our terminology, this means f is invariant under a set of transformations $\mathcal{R}$. We write this formally as

$$\mathbf{S}f = f \,\forall \mathbf{S} \in \mathcal{R}. \tag{3.2}$$

For the theories of interest here, $\mathcal{R}$ consists of all combinations of rotations and translations.

We take as a basic requirement on symmetry transformations the demand that $\mathbf{T}f$ should also be frame indifferent viz

$$\mathbf{S}\mathbf{T}f = \mathbf{T}f \,\forall \mathbf{S} \in \mathcal{R}.$$

†We note that this extension is not in any way suggested in [1]. It is done here with the overall aim of unifying various theories of symmetry.

‡Some liquid crystals exhibit [see 8] symmetry under reversal of sign of $\mathbf{n}$ and do not have symmetry under reflections.

§See the special case (2.3) on how $\mathbf{T}$ can be defined.

For this, it is sufficient that $T^{-1}ST \in \mathfrak{R}\, \forall S \in \mathfrak{R}$. In this study, we will assume this. Then given $S \in \mathfrak{R}$, $ST = T\tilde{S}$ for some $\tilde{S} \in \mathfrak{R}$. Therefore,

$$\begin{aligned} STf &= T\tilde{S}f \\ &= Tf \end{aligned}$$

from (3.2), which is what we want. A similar reasoning applied to T^{-1} makes us require that $TST^{-1} \in \mathfrak{R}\, \forall S$. So, for preservation of frame indifference, we require

$$T^{-1}ST, TST^{-1} \in \mathfrak{R}\, \forall S \in \mathfrak{R}. \tag{3.3}$$

An equivalent statement is that $T^{-1}\, \mathfrak{R} T = \mathfrak{R}$.

In writing these equations, we have assumed that every element S can act on both the domain and range of T. This is so for common materials.

Our intention is to use condition (3.3) to delimit our set of transformations $\mathfrak{D}$ for a class of materials which includes elastic materials and liquid crystals. To do this, we shall need to find all the automorphisms of $\mathfrak{R}$. Why this is necessary will be clear in Section 3(b).

For now, let G be the set consisting of all pairs $(\mathbf{R}, \mathbf{c})$ where $\mathbf{c} \in \mathbf{R}^3$ and $\mathbf{R}$ is a 3×3 rotation matrix, i.e.

$$\mathbf{R}\mathbf{R}^T = \mathbf{1}, \det \mathbf{R} = 1.$$

G is a group under the "product" operation defined by

$$(\mathbf{R}_1, \mathbf{c}_1)(\mathbf{R}_2, \mathbf{c}_2) = (\mathbf{R}_1\mathbf{R}_2, \mathbf{R}_1\mathbf{c}_2 + \mathbf{c}_1).$$

Lemma 3.1. *Let* $\tau : G \to G$ *be an automorphism*† *of* G. *Then, there exists a rotation* $\mathbf{R}_0$, *and a non-zero real number* $\alpha \ni$

$$(\mathbf{1}, \mathbf{z}) \overset{\tau}{\mapsto} (\mathbf{1}, \alpha \mathbf{R}_0 \mathbf{z})\, \forall \mathbf{z} \in \mathbf{R}^3,$$

$$(\mathbf{R}, \mathbf{o}) \overset{\tau}{\mapsto} (\mathbf{R}_0 \mathbf{R} \mathbf{R}_0^T, \mathbf{c}(\mathbf{R}))$$

for all rotations $\mathbf{R}$, *where* $\mathbf{c}$, *as indicated, depends on* $\mathbf{R}$.

Proof. For $x \in G$, let $C_G(x)$ be the centraliser of x, and $\mathscr{C}_G(x)$ be the conjugacy class of x in G, i.e.

$$C_G(x) = \{y \in G \mid xy = yx\},$$
$$\mathscr{C}_G(x) = \{yxy^{-1} \mid y \in G\}.$$

It can be verified that

$$\begin{aligned} \mathscr{C}_G((\mathbf{1}, \mathbf{z})) &= \{(\mathbf{1}, \mathbf{R}\mathbf{z}) \mid \mathbf{R}, \text{a rotation}\}, \\ &= \{(\mathbf{1}, \mathbf{w}) \mid |\mathbf{w}| = |\mathbf{z}|\}. \end{aligned} \tag{3.4}$$

We observe that

$$\mathscr{C}_G((\mathbf{1}, \mathbf{z})) \subset C_G((\mathbf{1}, \mathbf{z})). \tag{3.5}$$

We shall now prove that property (3.5) distinguishes $(\mathbf{1}, \mathbf{z})$ from any other element $(\mathbf{R}, \mathbf{c})$ with $\mathbf{R} \neq \mathbf{1}$.

†An automorphism of G is a 1–1 function of G onto G which preserves the group operation.

Let us consider $(\hat{\mathbf{R}}, \hat{\mathbf{c}})$ where $\hat{\mathbf{R}} \neq 1$ and $\hat{\mathbf{R}}\hat{\mathbf{e}} = \hat{\mathbf{e}}$; and let $\mathbf{R}$ be a rotation such that

$$\begin{aligned} \mathbf{R}\hat{\mathbf{e}} &\neq \pm\hat{\mathbf{e}}, \\ \hat{\mathbf{e}} \cdot \mathbf{R}\hat{\mathbf{e}} &\neq 0. \end{aligned} \tag{3.6}$$

Such $\mathbf{R}$ exists; we can, e.g. choose the rotation with axis perpendicular to $\hat{\mathbf{e}}$ and with angle of rotation 60°.

Now, if

$$(\mathbf{R}, \mathbf{o})(\hat{\mathbf{R}}, \hat{\mathbf{c}})(\mathbf{R}^T, \mathbf{o}) \in C_G((\hat{\mathbf{R}}, \hat{\mathbf{c}})),$$

then

$$(\mathbf{R}\hat{\mathbf{R}}\mathbf{R}^T)\hat{\mathbf{R}} = \hat{\mathbf{R}}(\mathbf{R}\hat{\mathbf{R}}\mathbf{R}^T),$$

which would imply that

$$\mathbf{R}\hat{\mathbf{R}}\mathbf{R}^T\hat{\mathbf{e}} = \hat{\mathbf{R}}(\mathbf{R}\hat{\mathbf{R}}\mathbf{R}^T\hat{\mathbf{e}}),$$

or

$$\mathbf{R}\hat{\mathbf{R}}\mathbf{R}^T\hat{\mathbf{e}} = \pm\hat{\mathbf{e}}. \tag{3.7}$$

If we take the plus sign in (3.7), we have that

$$\begin{aligned} &\hat{\mathbf{R}}(\mathbf{R}^T\hat{\mathbf{e}}) = \mathbf{R}^T\hat{\mathbf{e}}, \\ &\Rightarrow \mathbf{R}^T\hat{\mathbf{e}} = \pm\hat{\mathbf{e}} \end{aligned}$$

which contradicts the choice of $\mathbf{R}$ in (3.6). On the other hand, if we take the minus sign in (3.7), then

$$\begin{aligned} &\hat{\mathbf{R}}(\mathbf{R}^T\hat{\mathbf{e}}) = -\mathbf{R}^T\hat{\mathbf{e}}, \\ &\Rightarrow \hat{\mathbf{e}} \cdot \hat{\mathbf{R}}\mathbf{R}^T\hat{\mathbf{e}} = -\hat{\mathbf{e}} \cdot \mathbf{R}^T\hat{\mathbf{e}}, \\ &\Rightarrow (\mathbf{R}\hat{\mathbf{e}}) \cdot \hat{\mathbf{e}} = -(\mathbf{R}\hat{\mathbf{e}}) \cdot \hat{\mathbf{e}}, \end{aligned}$$

or

$$(\mathbf{R}\hat{\mathbf{e}}) \cdot \hat{\mathbf{e}} = 0,$$

which again contradicts (3.6). Hence

$$(\mathbf{R}, \mathbf{o})(\hat{\mathbf{R}}, \hat{\mathbf{c}})(\mathbf{R}^T, \mathbf{o}) \notin C_G((\hat{\mathbf{R}}, \hat{\mathbf{c}})), \text{ so, } \mathscr{C}_G((\hat{\mathbf{R}}, \hat{\mathbf{c}})) \not\subset C_G((\hat{\mathbf{R}}, \hat{\mathbf{c}})) \tag{3.8}$$

for $\hat{\mathbf{R}} \neq 1$.

Let

$$(1, \mathbf{z}) \overset{\tau}{\mapsto} (\bar{\mathbf{R}}_z, \bar{\mathbf{c}}_z)$$

and let $(\bar{\mathbf{R}}, \bar{\mathbf{c}})$ be an arbitrary element in G. Since τ is onto, $\exists\, (\mathbf{R}, \mathbf{c}) \ni \tau((\mathbf{R}, \mathbf{c})) = (\bar{\mathbf{R}}, \bar{\mathbf{c}})$. Since by (3.5), $(\mathbf{R}, \mathbf{c})(1, \mathbf{z})(\mathbf{R}, \mathbf{c})^{-1}$ commutes with $(1, \mathbf{z})$, and since τ is a homomorphism, $(\bar{\mathbf{R}}, \bar{\mathbf{c}})(\bar{\mathbf{R}}_z, \bar{\mathbf{c}}_z)(\bar{\mathbf{R}}, \bar{\mathbf{c}})^{-1}$ commutes with $(\bar{\mathbf{R}}_z, \bar{\mathbf{c}}_z)$. But $(\bar{\mathbf{R}}, \bar{\mathbf{c}})$ is arbitrary. Therefore, using (3.8), we conclude that

$$\bar{\mathbf{R}}_z = 1.$$

Hence

$$(1, \mathbf{z}) \overset{\tau}{\mapsto} (1, \bar{\mathbf{c}}_z) \forall \mathbf{z} \in \mathbf{R}^3. \tag{3.9}$$

Let us define $\theta: \mathbf{R}^3 \to \mathbf{R}^3$ by $\theta(\mathbf{z}) = \bar{\mathbf{c}}_z$. Then, θ satisfies:

$$\text{(i)}\ \theta(\mathbf{z}_1 + \mathbf{z}_2) = \theta(\mathbf{z}_1) + \theta(\mathbf{z}_2).$$

$$\text{(ii)}\ |\mathbf{z}| = |\mathbf{w}| \Rightarrow |\theta(\mathbf{z})| = |\theta(\mathbf{w})|.$$

This is because τ maps conjugates into conjugates.

$$\text{(iii)}\ \theta \text{ is 1–1 and onto.}$$

This follows from (3.9) and the fact that τ (and therefore τ^{-1}) is an automorphism.

If we define $\bar{\theta}$. $E_3 \to E_3$ by $\bar{\theta}(X_0 + \mathbf{z}) = X_0 + \theta(\mathbf{z})$, where E_3 is the euclidean space of dimension 3, and X_0 is a fixed element, we deduce from (i), (ii), (iii) above that $\bar{\theta}$ is 1–1 and onto, and preserves equality of distances. By Theorem 2.23 in[10] we have that

$$\bar{\theta}(X_0 + \mathbf{z}) = X_0 + \alpha \mathbf{R}_{0}\mathbf{z}$$

for some non-zero real number α, and some rotation $\mathbf{R}_0$. Hence

$$\theta(\mathbf{z}) = \alpha \mathbf{R}_0 \mathbf{z}$$

$$\therefore\ (1, \mathbf{z}) \overset{\tau}{\mapsto} (1, \alpha \mathbf{R}_0 \mathbf{z})$$

for some rotation $\mathbf{R}_0$ and some non-zero real number α as claimed by the lemma. Also, since

$$(1, \mathbf{R}\mathbf{w}) = (\mathbf{R}, \mathbf{o})(1, \mathbf{w})(\mathbf{R}^T, \mathbf{o}) \forall \mathbf{w},$$

we have that

$$(1, \alpha \mathbf{R}_0 \mathbf{R}\mathbf{w}) = (\bar{\mathbf{R}}, \bar{\mathbf{c}})(1, \alpha \mathbf{R}_0 \mathbf{w})(\bar{\mathbf{R}}^T, -\bar{\mathbf{R}}^T \bar{\mathbf{c}}) \forall \mathbf{w},$$

where, here

$$(\bar{\mathbf{R}}, \bar{\mathbf{c}}) \equiv \tau((\mathbf{R}, \mathbf{o})).$$

Hence

$$(1, \alpha \mathbf{R}_0 \mathbf{R}\mathbf{w}) = (1, \alpha \bar{\mathbf{R}} \mathbf{R}_0 \mathbf{w}) \forall \mathbf{w}.$$

This means that, for all $\mathbf{R}$, there is a function $\bar{\mathbf{c}}(\mathbf{R}) \ni$

$$(\mathbf{R}, \mathbf{o}) \overset{\tau}{\mapsto} (\mathbf{R}_0 \mathbf{R} \mathbf{R}_0{}^T, \bar{\mathbf{c}}(\mathbf{R})).$$

QED

Lemma 3.2. *Let $\tau: G \to G$ be an automorphism of G such that for every rotation R,*

$$(\mathbf{R}, \mathbf{o}) \overset{\tau}{\mapsto} (\bar{\mathbf{R}}, \bar{\mathbf{c}}(\mathbf{R}))$$

where $\bar{\mathbf{c}}(\mathbf{R})$ *is* **normal** *to the axis of* $\bar{\mathbf{R}}$.

Then, there exist a vector **d** (*independent of* **R**), *and a rotation* $\mathbf{R}_0$ *such that*

$$(\mathbf{R}, \mathbf{o}) \overset{T}{\mapsto} (\mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T, \mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T\mathbf{d}-\mathbf{d}).$$

Proof. In view of Lemma 3.1, all we need do is produce the vector **d**.

Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in R^3, and let $\mathbf{R}^z\pi$ denote the half-turn with axis parallel to z. Using Lemma 3.1, we realize that

$$(\mathbf{R}^e\pi, \mathbf{o}) \overset{T}{\mapsto} (\mathbf{R}_0\mathbf{R}^e\pi\mathbf{R}_0{}^T, \lambda_1\mathbf{R}_0\mathbf{e} + \lambda_2\mathbf{R}_0\mathbf{f} + \lambda_3\mathbf{R}_0\mathbf{g}),$$

$$(\mathbf{R}^f\pi, \mathbf{o}) \overset{T}{\mapsto} (\mathbf{R}_0\mathbf{R}^f\pi\mathbf{R}_0{}^T, \mu_1\mathbf{R}_0\mathbf{e} + \mu_2\mathbf{R}_0\mathbf{f} + \mu_3\mathbf{R}_0\mathbf{g}),$$

for some $\lambda_i, \mu_i, i = 1, 2, 3$. By hypothesis, $\lambda_1 = \mu_2 = 0$. Since $\mathbf{R}^e\pi\mathbf{R}^f\pi = \mathbf{R}^f\pi\mathbf{R}^e\pi = \mathbf{R}^g\pi$, it follows that $\lambda_3 = \mu_3$. It can now be verified that

$$\lambda_1\mathbf{R}_0\mathbf{e} + \lambda_2\mathbf{R}_0\mathbf{f} + \lambda_3\mathbf{R}_0\mathbf{g} = \mathbf{R}_0\mathbf{R}^e\pi\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d},$$

$$\mu_1\mathbf{R}_0\mathbf{e} + \mu_2\mathbf{R}_0\mathbf{f} + \mu_3\mathbf{R}_0\mathbf{g} = \mathbf{R}_0\mathbf{R}^f\pi\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d},$$

where

$$\mathbf{d} = -\frac{1}{2}(\mu_1\mathbf{R}_0\mathbf{e} + \lambda_2\mathbf{R}_0\mathbf{f} + \lambda_3\mathbf{R}_0\mathbf{g}). \tag{3.10}$$

We therefore have that

$$(\mathbf{R}^e\pi, \mathbf{o}) \overset{T}{\mapsto} (\mathbf{R}_0\mathbf{R}^e\pi\mathbf{R}_0{}^T, \mathbf{R}_0\mathbf{R}^e\pi\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d}),$$

$$(\mathbf{R}^f\pi, \mathbf{o}) \overset{T}{\mapsto} (\mathbf{R}_0\mathbf{R}^f\pi\mathbf{R}_0{}^T, \mathbf{R}_0\mathbf{R}^f\pi\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d}), \tag{3.11}$$

and, hence,

$$(\mathbf{R}^g\pi, \mathbf{o}) \overset{T}{\mapsto} (\mathbf{R}_0\mathbf{R}^g\pi\mathbf{R}_0{}^T, \mathbf{R}_0\mathbf{R}^g\pi\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d}).$$

Suppose $\mathbf{R}_1, \mathbf{R}_2$ are two rotations such that

$$\mathbf{R}_1\mathbf{e} = \mathbf{R}_2\mathbf{e} = \mathbf{e}.$$

Then, $(\mathbf{R}_1, \mathbf{o})$, $(\mathbf{R}_2, \mathbf{o})$, $(\mathbf{1}, \mathbf{e})$ commute with each other. This means that their images $(\bar{\mathbf{R}}_1, \bar{\mathbf{c}}_1)$, $(\bar{\mathbf{R}}_2, \bar{\mathbf{c}}_2)$, and $(\mathbf{1}, \bar{\mathbf{c}}_e)$ respectively commute. This implies that

$$\bar{\mathbf{R}}_1\bar{\mathbf{c}}_2 + \bar{\mathbf{c}}_1 = \bar{\mathbf{R}}_2\bar{\mathbf{c}}_1 + \bar{\mathbf{c}}_2, \bar{\mathbf{R}}_1\bar{\mathbf{c}}_e = \bar{\mathbf{R}}_2\bar{\mathbf{c}}_e = \bar{\mathbf{c}}_e. \tag{3.12}$$

Since $\bar{\mathbf{c}}_1$ is perpendicular to axis of $\bar{\mathbf{R}}_1$ by hypothesis, $\exists\, \mathbf{m}$, unique up to a component along axis of $\bar{\mathbf{R}}_1$ (i.e. parallel to $\bar{\mathbf{c}}_e$) $\ni$

$$\bar{\mathbf{c}}_1 = \bar{\mathbf{R}}_1\mathbf{m} - \mathbf{m}.$$

From this and (3.12), it follows that

$$\bar{\mathbf{R}}_2\bar{\mathbf{c}}_1 = \bar{\mathbf{R}}_2\bar{\mathbf{R}}_1\mathbf{m} - \bar{\mathbf{R}}_2\mathbf{m},$$

$$\bar{\mathbf{R}}_1(\bar{\mathbf{R}}_2\mathbf{m} - \bar{\mathbf{c}}_2) - (\bar{\mathbf{R}}_2\mathbf{m} - \bar{\mathbf{c}}_2) = \bar{\mathbf{c}}_1.$$

Therefore,

$$\bar{\mathbf{R}}_2\mathbf{m} - \bar{\mathbf{c}}_2 = \mathbf{m} + k\bar{\mathbf{c}}_e,$$
$$(\bar{\mathbf{R}}_2\mathbf{m} - \mathbf{m}) - \bar{\mathbf{c}}_2 = k\bar{\mathbf{c}}_e.$$

Since the left hand side of the latter is perpendicular to $\bar{\mathbf{c}}_e$, which from (3.12) is parallel to axis of $\bar{\mathbf{R}}_2$, we deduce that $k = 0$.

$$\therefore \quad \bar{\mathbf{R}}_2\mathbf{m} - \mathbf{m} = \bar{\mathbf{c}}_2 \text{ if } \bar{\mathbf{R}}_1\mathbf{m} - \mathbf{m} = \bar{\mathbf{c}}_1.$$

This observation with the result in (3.11) imply that

$$(\mathbf{R}, \mathbf{o}) \overset{\tau}{\mapsto} (\mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T, \mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d}) \tag{3.13}$$

if the axis of $\mathbf{R}$ is parallel to $\mathbf{e}$, $\mathbf{f}$ or $\mathbf{g}$.

We shall now prove that the same $\mathbf{d}$ works for $\mathbf{R}$ with axis not parallel to $\mathbf{e}$, $\mathbf{f}$ or $\mathbf{g}$. If we assume that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is a right-handed system, it is not difficult to see that every rotation $\mathbf{R}$ can be written as

$$\mathbf{R} = \mathbf{R}^g\mathbf{R}^f\mathbf{R}^e \tag{3.14}$$

for some $\mathbf{R}^g$, $\mathbf{R}^f$, $\mathbf{R}^e$ where $\mathbf{R}^u$ denotes a rotation with axis parallel to $\mathbf{u}$. Indeed, if

$$\mathbf{R}\mathbf{e} = \sin\theta\cos\psi\,\mathbf{e} + \sin\theta\sin\psi\,\mathbf{f} + \cos\theta\mathbf{g},$$

and if we set

$$[\mathbf{R}^g] \equiv \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$[\mathbf{R}^f] \equiv \begin{bmatrix} \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & \end{bmatrix},$$

it is true that

$$(\mathbf{R}^{fT}\mathbf{R}^{gT}\mathbf{R})\mathbf{e} = (\mathbf{R}^{fT}\mathbf{R}^{gT})\mathbf{R}\mathbf{e} = \mathbf{e},$$

and so, $\mathbf{R}^{fT}\mathbf{R}^{gT}\mathbf{R} = \mathbf{R}^e$ for some $\mathbf{R}^e$.
Therefore, (3.14) holds. From (3.13) and (3.14), we conclude that

$$(\mathbf{R}, \mathbf{o}) \overset{\tau}{\mapsto} (\mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T, \mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d})$$

where $\mathbf{d}$ is as given in (3.10).

QED

We now define a metric d on G by

$$d((\mathbf{R}_1, \mathbf{c}_1), (\mathbf{R}_2, \mathbf{c}_2)) = [\mathrm{t}((\mathbf{R}_1 - \mathbf{R}_2)(\mathbf{R}_1{}^T - \mathbf{R}_2{}^T)) + |\mathbf{c}_1 - \mathbf{c}_2|^2]^{1/2}.$$

If $(\mathbf{R}, \mathbf{c})$ is considered as an element in R^{12}, d is the Euclidean metric. If τ is a continuous automorphism of G, it must preserve compactness as well as the group operation. Therefore, if

$$(\mathbf{R}, \mathbf{o}) \stackrel{\tau}{\mapsto} (\bar{\mathbf{R}}, \bar{\mathbf{c}}), \tag{3.15}$$

$\bar{\mathbf{c}}$ must be normal to the axis of $\bar{\mathbf{R}}$. To see this clearly, note that

$$d((\mathbf{R}, \mathbf{o})^n, (1, \mathbf{o})) \leq 2\sqrt{3}$$

for all positive integers n and rotations $\mathbf{R}$, whereas

$$d((\bar{\mathbf{R}}, \bar{\mathbf{c}})^n, (1, \mathbf{o})) \longrightarrow \infty \text{ as } n \to \infty \text{ if } \bar{\mathbf{c}}$$

is not normal to axis of $\bar{\mathbf{R}}$. Since $(\mathbf{R}, \mathbf{c}) = (1, \mathbf{c})(\mathbf{R}, \mathbf{o}) \forall \mathbf{R}, \mathbf{c}$, Lemmas 3.1, and 3.2, and statement (3.15) yield the following

Theorem. Let τ: $G \to G$ *be a continuous automorphism of G. Then*

$$\tau: (\mathbf{R}, \mathbf{c}) \mapsto (\mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T, \alpha\mathbf{R}_0\mathbf{c} + \mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d})$$

where $\mathbf{R}_0$ *is some rotation,* $\mathbf{d}$ *is a vector in* $\mathbf{R}^3$ *and* α *is a non-zero real number.*

(b) *Application of theorem*

We shall use the theorem above in the following context:

As remarked in Section 3(a), for a constitutive equation (3.1), $\mathbf{a}$ may be any quantity. The special case discussed in Section 2 takes $\mathbf{a}$ to be the energy density W. Let us consider (3.1). Like (2.3), a symmetry transformation $\mathbf{T}$ in this case is also presumed to be induced by a transformation $\mathbf{T}'$ of the generalized coordinates, and $\mathbf{T}'$ is assumed to satisfy the analog of (2.2). Let us use the notation:

$$\hat{\mathbf{T}} \equiv (\mathbf{T}_1', \mathbf{T}_2'),$$

and let

$$\begin{aligned} u &= X \in \mathbf{R}^3, \\ \mathbf{v} &= (\mathbf{n}, \mathbf{x}) \in \mathbf{R}^3 \times \mathbf{R}^3, \\ \bar{\mathbf{v}} &\equiv (\bar{\mathbf{n}}, \bar{\mathbf{x}}) = (\mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}), \mathbf{T}'_{12}(\mathbf{n}, \mathbf{x})). \end{aligned}$$

The collection $\mathscr{R}'$ in this case is the set of all maps $\mathbf{S}' \ni$

$$\mathbf{S}': (\mathbf{n}, \mathbf{x}, X, \mathbf{a}) \mapsto (\mathbf{R}\mathbf{n}, \mathbf{R}\mathbf{x} + \mathbf{c}, X, \mathbf{S}\mathbf{a}) \forall (\mathbf{n}, \mathbf{x}, X) \in \mathcal{O} \tag{3.16}$$

where $\mathbf{R}$ is any rotation and, $\mathbf{c}$ is any vector in $\mathbf{R}^3$. $\mathbf{S}\mathbf{a}$ is normally inferred from what $\mathbf{a}$ is. In any case, usually $\mathscr{R}'$ forms a group.

We shall assume that $\mathcal{O}$ is closed under every element in $\hat{\mathscr{R}}$, and that (3.3) is satisfied when $\mathbf{T}$, $\mathbf{S}$ are replaced by the maps $\mathbf{T}'$, $\mathbf{S}'$, i.e.

$$\mathbf{T}'^{-1}\mathbf{S}'\mathbf{T}', \mathbf{T}'\mathbf{S}'\mathbf{T}'^{-1} \in \mathscr{R}' \forall \mathbf{S}' \in \mathscr{R}'. \tag{3.17}$$

Condition (3.17) implies

$$\left.\begin{aligned} \mathbf{T}'_{11}(\mathbf{R}\mathbf{n}, \mathbf{R}\mathbf{x} + \mathbf{c}, X) &= \bar{\mathbf{R}}\mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}, X), \\ \mathbf{T}'_{12}(\mathbf{R}\mathbf{n}, \mathbf{R}\mathbf{x} + \mathbf{c}, X) &= \bar{\mathbf{R}}\mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X) + \bar{\mathbf{c}}, \\ T_2'(\mathbf{R}\mathbf{n}, \mathbf{R}\mathbf{x} + \mathbf{c}, X) &= T_2'(\mathbf{n}, \mathbf{x}, X) \end{aligned}\right\} \tag{3.18}$$

$\forall(\mathbf{n}, \mathbf{x}, X) \in \mathcal{O}$ where $\bar{\mathbf{R}} = \bar{\mathbf{R}}(\mathbf{R}, \mathbf{c}), \bar{\mathbf{c}} = \bar{\mathbf{c}}(\mathbf{R}, \mathbf{c})$ for the fixed $\mathbf{T}'$. For a fixed $(\mathbf{n}, \mathbf{x}, X) \in \mathcal{O}$, if $\{(\mathbf{R}_k, \mathbf{c}_k)\}$ is a sequence such that

$$(\mathbf{R}_k, \mathbf{c}_k) \longrightarrow (\mathbf{R}_0, \mathbf{c}_0) \text{ as } k \to \infty,$$

then

$$\bar{\mathbf{R}}(\mathbf{R}_k, \mathbf{c}_k)\mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}, X) \longrightarrow \bar{\mathbf{R}}(\mathbf{R}_0, \mathbf{c}_0)\mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}, X);\ \bar{\mathbf{R}}(\mathbf{R}_k, \mathbf{c}_k)\mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X)$$

$$+ \bar{\mathbf{c}}(\mathbf{R}_k, \mathbf{c}_k) \longrightarrow \bar{\mathbf{R}}(\mathbf{R}_0, \mathbf{c}_0)\mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X) + \bar{\mathbf{c}}(\mathbf{R}_0, \mathbf{c}_0) \text{ as}$$

$k \to \infty$. This is deduced from the fact that $\mathbf{T}'$ is continuous in $\mathbf{n}, \mathbf{x}$. The metric we are using on $\{(\mathbf{R}, \mathbf{c})\}$ is the Euclidean metric on $\mathbf{R}^{12}$ which was introduced earlier in this section.

Since $\mathcal{O}$ is closed under every $\mathbf{S} \in \mathcal{R}$, $\mathbf{T}'_{12}(\mathcal{O}) = \mathbf{R}^3$. Therefore, $\exists(\bar{\mathbf{n}}, \bar{\mathbf{x}}, \bar{X}) \ni \mathbf{T}'_{12}(\bar{\mathbf{n}}, \bar{\mathbf{x}}, \bar{X}) = \mathbf{0}$. If we put this in the last equation, we deduce, first, that as $k \to \infty, \bar{\mathbf{c}}(\mathbf{R}_k, \mathbf{c}_k) \to \bar{\mathbf{c}}(\mathbf{R}_0, \mathbf{c}_0)$, and then, using other elements in $\mathcal{O}$, we infer that $\bar{\mathbf{R}}(\mathbf{R}_k, \mathbf{c}_k) \to \bar{\mathbf{R}}(\mathbf{R}_0, \mathbf{c}_0)$. Hence, for the fixed $\mathbf{T}'$, the induced mapping

$$(\mathbf{R}, \mathbf{c}) \mapsto (\bar{\mathbf{R}}(\mathbf{R}, \mathbf{c}), \bar{\mathbf{c}}(\mathbf{R}, \mathbf{c}))$$

is a homeomorphism of G. It also preserves group operation since, by definition, $\mathbf{T}'\mathbf{S}' = \bar{\mathbf{S}}'\mathbf{T}'$. By the theorem above,

$$\begin{aligned} \bar{\mathbf{R}}(\mathbf{R}, \mathbf{c}) &= \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T \\ \bar{\mathbf{c}}(\mathbf{R}, \mathbf{c}) &= \alpha\mathbf{R}_0\mathbf{c} + \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{d} - \mathbf{d}, \end{aligned} \tag{3.19}$$

for some rotation $\mathbf{R}_0$, some vector $\mathbf{d}$ and some non-zero real number α.

From (3.18) and (3.19), we have

$$\begin{aligned} &\mathbf{T}'_{11}(\mathbf{R}\mathbf{n}, \mathbf{R}\mathbf{x} + \mathbf{c}, X) = \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}, X), \\ &\mathbf{T}'_{12}(\mathbf{R}\mathbf{n}, \mathbf{R}\mathbf{x} + \mathbf{c}, X) = \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X) + \alpha\mathbf{R}_0\mathbf{c} + \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{d} - \mathbf{d}, \\ &\qquad T'_2(\mathbf{n}, \mathbf{x}, X) = T'_2(|\mathbf{n}|, X)\ \forall(\mathbf{n}, \mathbf{x}, X) \in \mathcal{O}. \end{aligned} \tag{3.20}$$

By putting $\mathbf{R} = 1, \mathbf{c} = -\mathbf{x}$ in $(3.20)_1$, we conclude that

$$\mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}, X) = \mathbf{T}'_{11}(\mathbf{n}, \mathbf{0}, X) = \mathbf{T}'_{11}(\mathbf{n}, X),$$

and

$$\mathbf{T}'_{11}(\mathbf{R}\mathbf{n}, X) = \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{T}'_{11}(\mathbf{n}, X).$$

We choose $\mathbf{R}$ for which $\mathbf{R}\mathbf{n} = \mathbf{n}$ to obtain

$$\mathbf{T}'_{11}(\mathbf{n}, X) = \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{T}'_{11}(\mathbf{n}, X)$$

which then implies that there exists a scalar $\beta(\mathbf{n}, X)$ such that

$$\mathbf{T}'_{11}(\mathbf{n}, X) = \beta(\mathbf{n}, X)\mathbf{R}_0\mathbf{n}.$$

Here, β is uniquely determined unless $\mathbf{n} = \mathbf{0}$, when it can be any real number. Of course, $\mathbf{n} = \mathbf{0}$ need not occur in $\mathcal{O}$. Also, for $\mathbf{R}: \mathbf{R}\mathbf{n} = \mathbf{n}$,

$$\mathbf{T}'_{12}(\mathbf{n}, \mathbf{R}\mathbf{x} + \mathbf{c}, X) = \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X) + \alpha\mathbf{R}_0\mathbf{c} + \mathbf{R}_0\mathbf{R}\mathbf{R}_0^T\mathbf{d} - \mathbf{d}.$$

Thus

$$\mathbf{T}_{12}(\mathbf{n}, \mathbf{c}, X) = \mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T\mathbf{T}'_{12}(\mathbf{n}, \mathbf{0}, X) + \alpha\mathbf{R}_0\mathbf{c} + \mathbf{R}_0\mathbf{R}\mathbf{R}_0{}^T\mathbf{d} - \mathbf{d},$$

and

$$\mathbf{T}'_{12}(\mathbf{n}, \mathbf{0}, X) = \gamma(\mathbf{n}, X)\mathbf{R}_0\mathbf{n} - \mathbf{d}$$

where γ has the same properties as β. Combining the above equations, we get

$$\mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X) = \alpha\mathbf{R}_0\mathbf{x} + \gamma(\mathbf{n}, X)\mathbf{R}_0\mathbf{n} - \mathbf{d}.$$

Hence,

$$(\mathbf{n}, \mathbf{x}, X) \overset{\hat{T}}{\mapsto} ((\bar{\mathbf{n}}, \bar{\mathbf{x}}, \bar{X})$$

where

$$\begin{aligned}
\bar{\mathbf{n}} &\equiv \mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}, X) = \beta(\mathbf{n}, X)\mathbf{R}_0\mathbf{n},\\
\bar{\mathbf{x}} &\equiv \mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X) = \alpha\mathbf{R}_0\mathbf{x} + \gamma(\mathbf{n}, X)\mathbf{R}_0\mathbf{n} - \mathbf{d},\\
\bar{X} &\equiv T'_2(|\mathbf{n}|, X).
\end{aligned}$$

Using this form, we deduce easily that $\hat{\mathbf{T}}$ satisfies (3.18) if, and only if

$$\begin{aligned}
\bar{\mathbf{n}} &\equiv \mathbf{T}'_{11}(\mathbf{n}, \mathbf{x}, X) = \beta(|\mathbf{n}|, X)\mathbf{R}_0\mathbf{n},\\
\bar{\mathbf{x}} &\equiv \mathbf{T}'_{12}(\mathbf{n}, \mathbf{x}, X) = \alpha\mathbf{R}_0\mathbf{x} + \gamma(|\mathbf{n}|, X)\mathbf{R}_0\mathbf{n} - \mathbf{d},\\
\bar{X} &\equiv T'_2(\mathbf{n}, \mathbf{x}, X) = T'_2(|\mathbf{n}|, X).
\end{aligned} \tag{3.21}$$

This is the reduced form for $\hat{\mathbf{T}}$ which shall be used in Section 4. The functions β, γ, T'_2 which are well-defined except possibly when $\mathbf{n} = \mathbf{0}$ will be further restricted by the invertibility and differentiability requirements which are analogous to those in (2.2).

We note that arguments entirely similar to the ones we present here can be used when the body is 2 dimensional, i.e. $\mathbf{x} \in R^2$ or when the argument $\mathbf{n}$ is not present. The result for the latter is

$$\begin{aligned}
\bar{\mathbf{x}} &= \mathbf{T}'_1(\mathbf{x}, X) = \alpha\mathbf{R}_0\mathbf{x} - \mathbf{d},\\
\bar{X} &= T'_2(X).
\end{aligned} \tag{3.22}$$

4. SYMMETRY TRANSFORMATIONS FOR ELASTIC MATERIALS AND LIQUID CRYSTALS

In this section, we show how the symmetry transformations for the theories of elasticity, and liquid crystals which are quoted in Section 2 are derived.

Elasticity theory

For elasticity theory, the constitutive equation involving the energy density is of the form

$$W = \hat{W}\left(\frac{\partial \mathbf{x}}{\partial X}, X\right) \tag{4.1}$$

where $\mathbf{x} \in R^3$, $X \equiv (X^1, X^2, X^3)$ denotes a material point. It is clear that this fits into the general eqn (2.1). Let

$$\mathbf{T}':(\mathbf{x}, X, W)\mapsto\left(\mathbf{T}_1'(\mathbf{x}, X), T_2'(\mathbf{x}, X), (W + A(X))\left\|\frac{\partial X}{\partial \bar{X}}\right\|\right)\forall(\mathbf{x}, X)\in \mathcal{O}_1.$$

We now consider the requirement that T' preserves frame indifference.

Let $\mathcal{R}_1'$ be the collection of all S' where S' is as defined in (3.16) except that the argument **n** is dropped. As argued in Section 3, frame indifference is preserved if we require that (3.17) is satisfied, i.e. if

$$\mathbf{T}'\mathcal{R}_1'\mathbf{T}'^{-1} = \mathbf{T}'^{-1}\mathcal{R}_1'\mathbf{T}' = \mathcal{R}_1'.$$

By (3.22) and (2.3), such T' has the form

$$\begin{aligned} \mathbf{T}_1'(\mathbf{x}, X) &= \alpha\mathbf{R}_0\mathbf{x} + \mathbf{d}, \\ T_2'(\mathbf{x}, X) &= T_2'(X) = \bar{X}(X), \\ \bar{W} &= (W + A(X))\left\|\frac{\partial X}{\partial \bar{X}}\right\|. \end{aligned} \tag{4.2}$$

Here: T_2' is invertible, $\mathbf{R}_0$ is some rotation, α is some non-zero real number and **d** is some vector in $\mathbf{R}^3$. Let us restrict ourselves to T' for which T_2' maps a neighborhood of the material point X_0 into another neighborhood, and $T_2'(X_0) = X_0$ so that $X_0 = \bar{u}_0 = u_0$. The set of all such T forms a group under composition, two maps being considered the same if they assume equal values for all arguments $(\mathbf{x}, X, W)$ for which X belongs to a neighborhood of X_0. The constitutive eqn (4.1) is invariant under (4.2) at point X_0 if, and only if,

$$\left(\hat{W}\left(\frac{\partial \mathbf{x}}{\partial X}, X_0\right) + A(X_0)\right)\left\|\frac{\partial X}{\partial \bar{X}}\right\| = \hat{W}\left(\alpha\mathbf{R}_0\frac{\partial \mathbf{x}}{\partial X}\frac{\partial X}{\partial \bar{X}}, X_0\right)$$

for every $\frac{\partial \mathbf{x}}{\partial X}$,

$$\Leftrightarrow \hat{W}\left(\frac{\partial \mathbf{x}}{\partial X}, X_0\right) = \hat{W}\left(\alpha\mathbf{R}_0\frac{\partial \mathbf{x}}{\partial X}\frac{\partial X}{\partial \bar{X}}, X_0\right)\left\|\frac{\partial \bar{X}}{\partial X}\right\| - A(X_0).$$

As used earlier, $\left|\frac{\partial \bar{X}}{\partial X}\right|$ represents $\left(\det\frac{\partial \bar{X}}{\partial X}\right)$ and $\left\|\frac{\partial \bar{X}}{\partial X}\right\| = \left|\det\left(\frac{\partial \bar{X}}{\partial X}\right)\right|$.

It is most commonly assumed in the literature that $A = 0$, $\alpha = \pm 1$. However, Truesdell[7] considered transformations that correspond to putting $\alpha = 1$, $\left\|\frac{\partial \bar{X}}{\partial X}\right\| = 1$ in (4.2) but allowed for A not being zero and Wang in[4] gave some examples of materials having some symmetry transformations with $A \neq 0$. Gurtin and Williams[11] argued that, if the symmetry transformations to be considered are equivalent to those obtained by setting $\alpha = 1$, $A = 0$ in (4.2), then by thermomechanical reasoning, $\left\|\frac{\partial \bar{X}}{\partial X}\right\|$ must also be set equal to 1; Liu's[12] paper is along the same line although his definition of symmetry is a little different.

Though transformations with $|\alpha| = 1$ characterise the symmetry of many materials as shown in the references cited above, yet there are materials whose symmetry cannot be fully described by these transformations. For instance, let us find the group that *characterises* the symmetry of the ideal gas in the class of elastic materials. For an ideal gas which is always *under isothermal conditions*, the energy density is

$$\hat{W}\left(\frac{\partial \mathbf{x}}{\partial X}, X_0\right) = k\log\left\|\frac{\partial \mathbf{x}}{\partial X}\right\| + c \tag{4.3}$$

where $k > 0$, c is a constant. It can be verified that (4.3) is invariant under the following special class of elements of form (4.2):

$$
\begin{aligned}
\mathbf{T}_1'(\mathbf{x}, X) &= \alpha \mathbf{R}_0 \mathbf{x} + \mathbf{d}, \\
T_2'(\mathbf{x}, X) &= \bar{X}(X), \qquad (4.4)\\
\bar{W} &= W + 3k \log \left| \alpha \right|,
\end{aligned}
$$

where $\left\|\frac{\partial \bar{X}}{\partial X}\right\| = 1$. Conversely, as can be verified, within the class of elastic materials, the material that has (4.4) as its symmetry group is the ideal gas (4.3).

This example shows one feature of not restricting symmetry transformations to those with $|\alpha| = 1$. Although (4.3) is invariant under all unimodular transformations (i.e. in this case $|\alpha| = 1$), it is by considering transformations with $|\alpha| \neq 1$ that we are able to characterise its symmetry.

We observe that there is no material known which is invariant under the transformations that arise from

$$
\begin{aligned}
\mathbf{T}_1'(\mathbf{x}, X) &= \alpha \mathbf{R}_0 \mathbf{x} + \mathbf{d}, \\
T_2'(\mathbf{x}. \mathrm{X}) &= \bar{X}(X), \\
\bar{W} &= W
\end{aligned}
$$

where $\left|\alpha \frac{\partial \bar{X}}{\partial X}\right| = |\alpha|^3 \neq 1$ since such a material would assume a constant value for all $\mathbf{F}$. This shows that we should not expect the whole of the symmetry group for a class to correspond to a material. The set of all transformations

$$\mathbf{T}: \left(\left(\frac{\partial \mathbf{x}}{\partial X}, X_0\right), W\right) \mapsto \left(\left(\alpha \mathbf{R}_0 \frac{\partial \mathbf{x}}{\partial X} \frac{\partial X}{\partial \bar{X}}, X_o\right), \left(W + A(X_0)\right) \left\|\frac{\partial X}{\partial \bar{X}}\right\|\right)$$

where $\alpha \left|\frac{\partial X}{\partial \bar{X}}\right| > 0$ constitute the group $\mathscr{S}_{01}$ from which symmetry groups can be chosen for materials within the class of elastic materials. The condition

$$\alpha \left|\frac{\partial X}{\partial \bar{X}}\right| > 0 \text{ merely ensures that } \left|\frac{\partial \mathbf{x}}{\partial X}\right| > 0 \text{ implies } \left|\frac{\partial \bar{\mathbf{x}}}{\partial \bar{X}}\right| > 0.$$

Liquid crystals

We shall now show that liquid crystals are special materials in the class of materials having constitutive equation

$$W = \hat{W}\left(\frac{\partial \mathbf{n}}{\partial X}, \frac{\partial \mathbf{x}}{\partial X}, \mathbf{n}, X\right) \qquad (4.5)$$

where $(\mathbf{n}, \mathbf{x}, X) \in \mathcal{O}_2 \subset \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$.

It was derived in Section 3b that the form (3.21) for $\mathbf{T}'$ ensures that frame indifference is preserved, i.e.

$$
\begin{aligned}
\bar{\mathbf{n}} &= \beta(|\mathbf{n}|, X) \mathbf{R}_0 \mathbf{n}, \\
\bar{\mathbf{x}} &= \propto \mathbf{R}_0 \mathbf{x} + \gamma(|\mathbf{n}|, X) \mathbf{R}_0 \mathbf{n} - \mathbf{d}, \qquad (4.6)\\
\bar{X} &= T_3'(|\mathbf{n}|, X).
\end{aligned}
$$

For liquid crystals, the state space is commonly considered to be

$$\left\{\left(\frac{\partial \mathbf{n}}{\partial X}, \frac{\partial \mathbf{x}}{\partial X}, \mathbf{n}, X_0\right) \middle| \mathbf{n} \cdot \mathbf{n} = 1, \left(\frac{\partial \mathbf{n}}{\partial X}\right)^T \mathbf{n} = 0, \left|\frac{\partial \mathbf{x}}{\partial X}\right| = 1\right\}.$$

As done for elastic materials, in order to discuss unusual symmetry of a point, we assume that $T_3(1, \cdot)$ maps a neighborhood of X_0 into a neighborhood of X_0. Since there is no element in $\mathcal{O}_2$ with $\mathbf{n}=\mathbf{0}$, β, γ are well-defined everywhere. Also, it makes no difference whether we set $|\mathbf{n}|=1$ in (4.6) before finding $\frac{\partial \bar{\mathbf{n}}}{\partial \bar{X}}, \frac{\partial \bar{\mathbf{x}}}{\partial \bar{X}}$ or after. So, we can take $\beta(|\mathbf{n}|, X) = \pm 1$; also, $\left|\frac{\partial \bar{\mathbf{x}}}{\partial \bar{X}}\right| = 1$ whenever $\left|\frac{\partial \mathbf{x}}{\partial X}\right| = 1$. The functions γ and T_3' must also reduce to functions of X only. The condition that $\left|\frac{\partial \bar{\mathbf{x}}}{\partial \bar{X}}\right| = 1$ whenever $\left|\frac{\partial \mathbf{x}}{\partial X}\right| = 1$ is therefore equivalent to

$$\left|\alpha \frac{\partial \mathbf{x}}{\partial X}(X_0) + \mathbf{n} \otimes \frac{\partial \gamma}{\partial X}(X_0) + \gamma \frac{\partial \mathbf{n}}{\partial X}(X_0)\right| = \left|\frac{\partial \bar{X}}{\partial X}\right| \tag{4.7}$$

whenever $\left|\frac{\partial \mathbf{x}}{\partial X}\right| = 1$ and for every $\frac{\partial \mathbf{n}}{\partial X}$ for which $\left(\frac{\partial \mathbf{n}}{\partial X}\right)^T \mathbf{n} = \mathbf{0}$.

With choice of $\frac{\partial \mathbf{n}}{\partial X} = \mathbf{0}$, and choice of diagonal matrices for $\frac{\partial \mathbf{x}}{\partial X}$ in (4.7), we conclude that $\frac{\partial \gamma}{\partial X}(X_0) = 0$. Further, if we choose

$$\mathbf{n}(X) = (\cos(X^2 - X_0^2), \sin(X^2 - X_0^2), 0),$$

and choose $\frac{\partial \mathbf{x}}{\partial X}$ diagonal in (4.7), we deduce that

$$\gamma(X_0) = 0.$$

Therefore, without loss in generality, if $\mathbf{T}' \in \mathcal{S}'_{02}$, it will be of the form

$$\begin{aligned} \bar{\mathbf{n}} &= \pm \mathbf{R}_0 \mathbf{n}, \\ \bar{\mathbf{x}} &= \alpha \mathbf{R}_0 \mathbf{x} - \mathbf{d}, \\ \bar{X} &= \bar{X}(X), \\ \bar{W} &= (W + A(X)) \left\| \frac{\partial X}{\partial \bar{X}} \right\|, \end{aligned} \tag{4.8}$$

where $\left\|\frac{\partial \bar{X}}{\partial X}\right\| = |\alpha|^3$.

Suppose a special material M is characterized by a group of symmetry $\mathcal{G}$ that contains *all* transformations of the form

$$\begin{aligned} \mathbf{T}': \bar{\mathbf{n}} &= \pm \mathbf{n}, \\ \bar{\mathbf{x}} &= \mathbf{x}, \\ \bar{X} &= \bar{X}(X), \\ \bar{W} &= W \left\| \frac{\partial X}{\partial \bar{X}} \right\| = W. \end{aligned}$$

Since we assume that elements in the domain of definition of W satisfies $\left|\frac{\partial \mathbf{x}}{\partial X}\right| = 1$, we easily infer that the energy density for M satisfies

$$\hat{W}\left(\frac{\partial \mathbf{n}}{\partial X}, \frac{\partial \mathbf{x}}{\partial X}, \mathbf{n}, X_0\right) = \hat{W}\left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}}, \frac{\partial \mathbf{x}}{\partial X}, \mathbf{n}, X_0\right)$$

$$= \hat{\hat{W}}\left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}}, \mathbf{n}, X_0\right) \tag{4.9}$$

$$= \hat{\hat{W}}\left(-\frac{\partial \mathbf{n}}{\partial \mathbf{x}}, -\mathbf{n}, X_0\right).$$

Of course, frame indifference will imply that

$$\hat{\hat{W}}\left(\mathbf{R}\frac{\partial \mathbf{n}}{\partial \mathbf{x}}\mathbf{R}^T, \mathbf{R}\mathbf{n}, X_0\right) = \hat{\hat{W}}\left(\frac{\partial \mathbf{n}}{\partial \mathbf{x}}, \mathbf{n}, X_0\right), \tag{4.10}$$

for all rotations. Let us assume that we can neglect X_0. If $\mathcal{G}$ contains no other element other then the ones already stated, examples of material M are provided by the cholesteric liquid crystals. However, if we assume that $\mathcal{G}$ also contains elements of the form

$$\bar{\mathbf{n}} = \mathbf{n},$$
$$\bar{\mathbf{x}} = \pm\mathbf{x},$$
$$\bar{X} = X,$$

then M has the symmetry of a nematic liquid crystal.

What we have proved is that the symmetry of a cholesteric liquid crystal in the class specified by (4.5) is characterized by the symmetry group consisting of all $\mathbf{T}'$:

$$\begin{aligned} \bar{\mathbf{n}} &= \pm\mathbf{R}_0\mathbf{n}, \\ \bar{\mathbf{x}} &= \mathbf{R}_0\mathbf{x} - \mathbf{d}, \\ \bar{X} &= \bar{X}(X), \\ \bar{W} &= W \end{aligned} \tag{4.11}$$

where $\left|\frac{\partial \bar{X}}{\partial X}\right| = 1$, and that that of a nematic liquid crystal is described by maps of the form

$$\begin{aligned} \bar{\mathbf{n}} &= \pm\mathbf{R}_0\mathbf{n}, \\ \bar{\mathbf{x}} &= \pm\mathbf{R}_0\mathbf{x} - \mathbf{d}, \\ \bar{X} &= \bar{X}(X), \\ \bar{W} &= W \end{aligned} \tag{4.12}$$

where $\left|\frac{\partial \bar{X}}{\partial X}\right| = 1$ and where the two ($\pm$) signs are not associated.

As remarked in Section 2, the set of transformations of material coordinates alone does not adequately account for the symmetry of liquid crystals in the class (4.5). Equation (2.6) follows by a reasoning similar to that in eqns (4.9) and (4.10). The material which has constitutive eqn (2.6) has a symmetry which corresponds to that of a fluid in elasticity.

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